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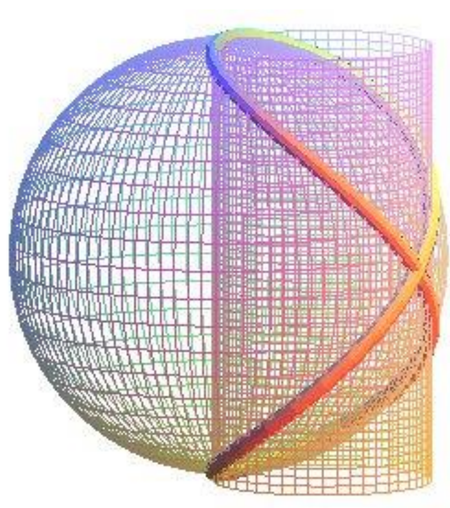
Curba lui Viviani



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Repere istorice

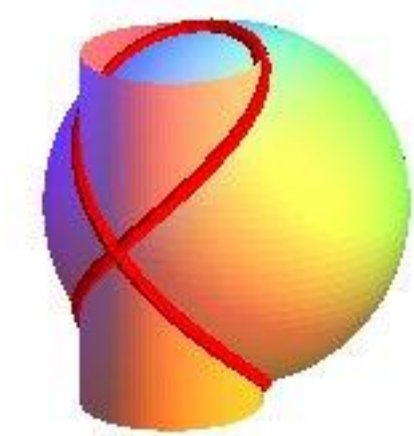
Vincenzo Viviani (5 Aprilie 1622- 22 Septembrie 1703) matematician si savant italian. Elevul lui Torricelli si discipol al lui Galileo Galilei.

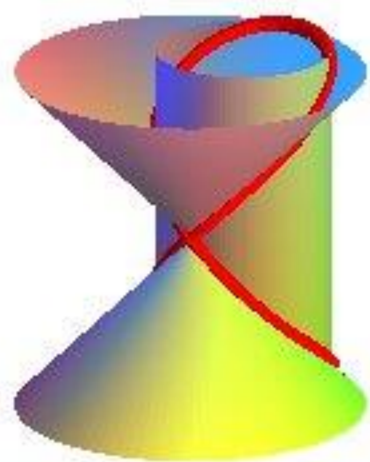
Nascut si crescut in Florenta el a studiat la o scoala Iezuita. Acolo, Marele Duce, Ferdinando II de Medici, i-a daruit o bursa pentru a-si cumpara cartile de matematica de care avea nevoie. A lucrat in domeniul fizicii si geometriei.

Exista un crater pe Luna denumit Viviani dupa numele matematicianului.

Aceasta curba a fost studiata de Viviani si Roberval in 1692. Numele initial, dat de Roberval, era de ciclo-cilindrica insa este mai cunoscuta sub numele de “fereastra lui Viviani”.

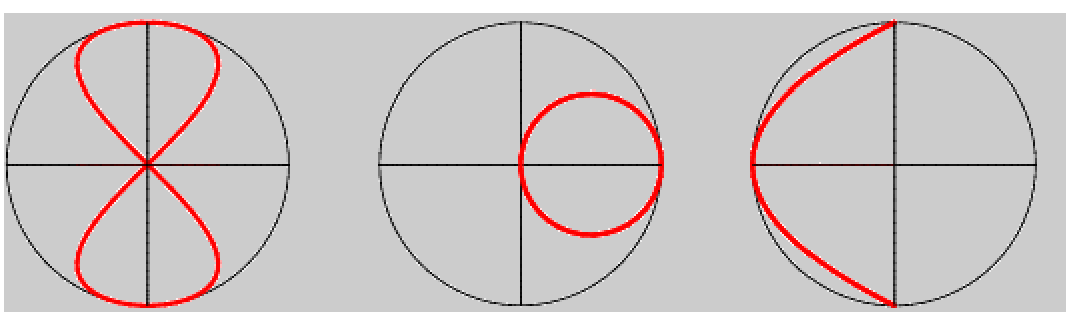
Curba lui Viviani este intersectia unei sfere de raza R si a unui cilindru de rotatie de diametru R a carei generatoare trece prin centrul sferei. Ea este de asemenea inclusa intr-un con de rotatie a carui axa este o generatoare a unui cilindru. Acestea se pot observa in figurile urmatoare.





Proiectiile pe *xOy*, *xOz* et *yOz* sunt respective un cerc(2), un arc de parabola(1) si lemniscata lui Gerono (3).

1 2 3



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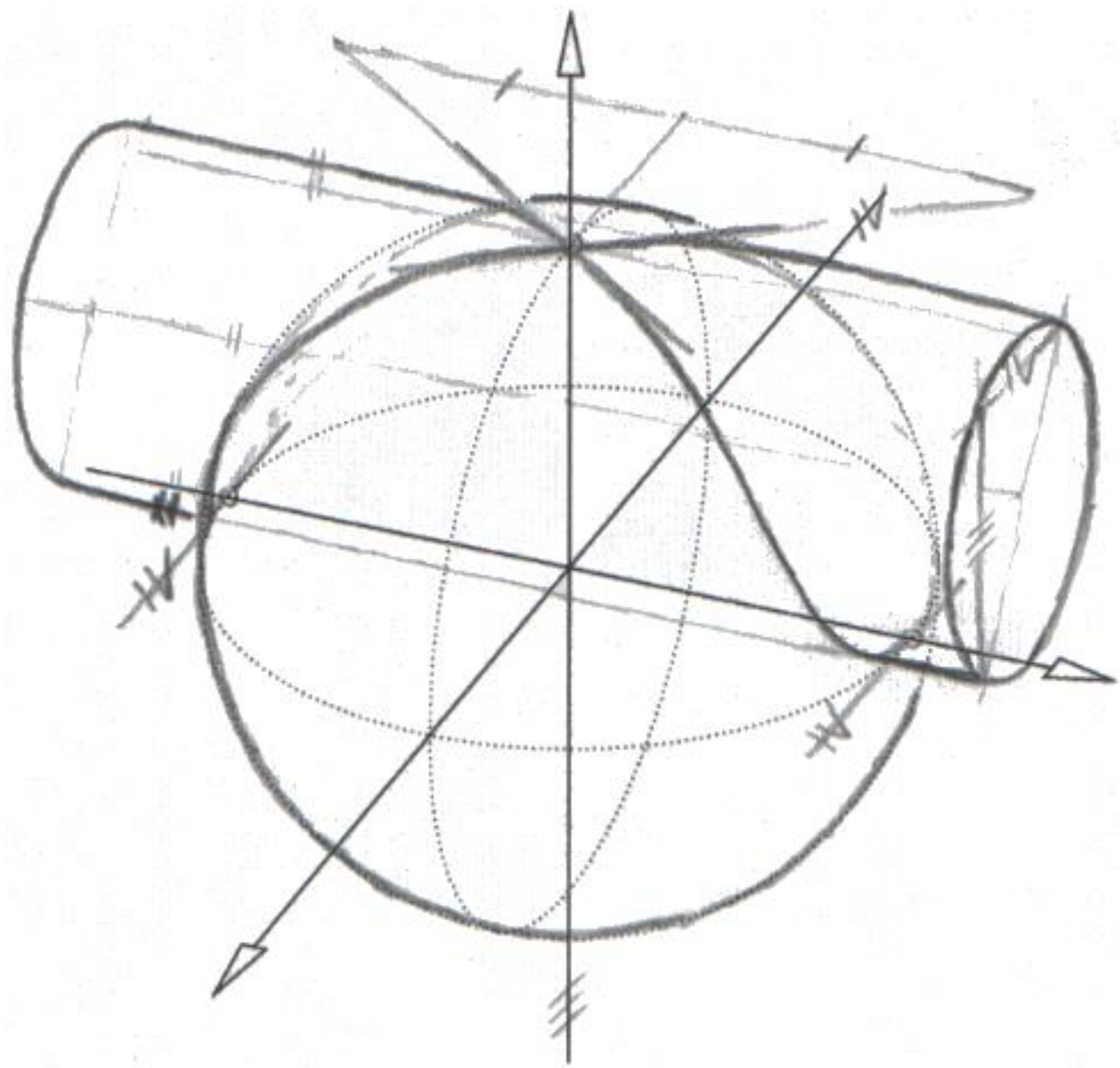
Pentru o mai buna intelegere curbei, prezentam in continuare o schita a celor mai sus mentionate.

Lumgimea totala este egala cu lungimea unei elispser de semiaxe R si R/2 data de o integrala eliptica, fiind aproximativ egala cu 4,844 R



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Studiul curbei

$c: \mathbb{R} \to \mathbb{E}_3$

$c(t) = \left(a + a\cos t,\ a\sin t,\ 2a\sin\frac{t}{2}\right)$

Curba în poziție generală

$\iff \{\dot{c}(t), \ddot{c}(t)\}$ sistem liniar independent

$\dot{c}(t) = \left(-a\sin t,\ a\cos t,\ a\cos\frac{t}{2}\right)$

$\ddot{c}(t) = \left(-a\cos t,\ -a\sin t,\ -\frac{a}{2}\sin\frac{t}{2}\right)$

$\iff \alpha \cdot \dot{c}(t) + \beta \cdot \ddot{c}(t) = O_3$

$\iff \alpha\left(-a\sin t,\ a\cos t,\ a\cos\frac{t}{2}\right) + \beta\left(-a\cos t,\ -a\sin t,\ -\frac{a}{2}\sin\frac{t}{2}\right) = 0$

$$\iff \begin{cases} -\alpha a\sin t - \beta a\cos t = 0 \\ \alpha a\cos t - \beta a\sin t = 0 \\ \alpha a\cos\frac{t}{2} - \beta\frac{a}{2}\sin\frac{t}{2} = 0 \Rightarrow \alpha = \frac{\beta}{2}\operatorname{tg}\frac{t}{2} = \frac{\beta}{2}\,\frac{1-\cos t}{\sin t} \end{cases}$$

$\alpha a\sin t + \beta a\cos t = 0 \ \Big|\cdot\frac{1}{a}, \quad a \neq 0$

$\alpha\sin t + \beta\cos t = 0$

$\Rightarrow \frac{\beta}{2}\,\frac{1-\cos t}{\cancel{\sin t}}\cdot\cancel{\sin t} + \beta\cos t = 0$

$\Rightarrow \frac{\beta}{2}(1-\cos t) + \beta\cos t = 0$

$\Rightarrow \beta\left(\frac{1-\cos t}{2} + \cos t\right) = 0$

$\Rightarrow \beta\,\frac{1+\cos t}{2} = 0 \Rightarrow \beta(1+\cos t) = 0,\ (\forall)t$

$\Rightarrow \beta = 0$

$\alpha\cos t - \beta\sin t = 0 \Rightarrow \alpha\cos t = 0,\ (\forall)t$

$\Rightarrow \alpha = 0$

$\Rightarrow \{\dot{c}(t), \ddot{c}(t)\}$ este sistem liniar independent

$\Rightarrow c(t)$ este curbă în poziție generală

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Curbura și torsiunea

curbura : $K_1 = \dfrac{\| \dot{c}(t) \times \ddot{c}(t) \|}{\| \dot{c}(t) \|^3}$

torsiunea : $K_2 = \dfrac{\det\left(\dot{c}(t), \ddot{c}(t), \dddot{c}(t) \right)}{\| \dot{c}(t) \times \ddot{c}(t) \|^2}$

$$\dot{c}(t) = \left(-a\sin t,\ a\cos t,\ a\cos\frac{t}{2} \right)$$
$$\ddot{c}(t) = \left(-a\cos t,\ -a\sin t,\ -\frac{a}{2}\sin\frac{t}{2} \right)$$
$$\dddot{c}(t) = \left(a\sin t,\ -a\cos t,\ -\frac{a}{4}\cos\frac{t}{2} \right)$$

$$\dot{c}(t) \times \ddot{c}(t) = \begin{vmatrix} -a\sin t & -a\cos t & i \\ a\cos t & -a\sin t & j \\ a\cos\frac{t}{2} & -\frac{a}{2}\sin\frac{t}{2} & k \end{vmatrix} = a^2 \begin{vmatrix} \sin t & -a\cos t & i \\ \cos t & -\sin t & j \\ \cos\frac{t}{2} & -\sin\frac{t}{2} & k \end{vmatrix}$$

$$\begin{vmatrix} \cos t & -\sin t \\ \cos\frac{t}{2} & -\frac{1}{2}\sin\frac{t}{2} \end{vmatrix} = -\frac{1}{2}\cos t \sin\frac{t}{2} + \sin t \cos\frac{t}{2}$$
$$2\sin\frac{t}{2}\cos\frac{t}{2}\cos\frac{t}{2}$$
$$= -\frac{1}{2}\left(2\cos^2\frac{t}{2} - 1 \right)\sin\frac{t}{2} + 2\sin\frac{t}{2}\cos^2\frac{t}{2}$$
$$= -\cos^2\frac{t}{2}\sin\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2} + 2\sin\frac{t}{2}\cos^2\frac{t}{2}$$
$$= \frac{1}{2}\sin\frac{t}{2} + \sin\frac{t}{2}\cos^2\frac{t}{2}$$
$$= \sin\frac{t}{2}\left(\frac{1}{2} + \cos^2\frac{t}{2} \right)$$
$$= \sin\frac{t}{2}\left(\frac{1}{2} + \frac{1+\cos t}{2} \right)$$
$$= \frac{1}{2}\sin\frac{t}{2}\left(\cos t + 2 \right)$$

$$-\begin{vmatrix} -\sin t & -\cos t \\ \cos\frac{t}{2} & -\frac{1}{2}\sin\frac{t}{2} \end{vmatrix} = -\left(\frac{1}{2}\sin t \sin\frac{t}{2} + \cos t \cos\frac{t}{2} \right) =$$
$$= -\left(\frac{1}{2} \cdot 2\sin\frac{t}{2}\cos\frac{t}{2}\sin\frac{t}{2} + \left(1 - 2\sin^2\frac{t}{2} \right)\cos\frac{t}{2} \right)$$
$$= -\left(\sin^2\frac{t}{2}\cos\frac{t}{2} + \cos\frac{t}{2} - 2\sin^2\frac{t}{2}\cos\frac{t}{2} \right)$$
$$= -\left(\cos\frac{t}{2} - \sin^2\frac{t}{2}\cos\frac{t}{2} \right) =$$
$$= -\cos\frac{t}{2}\left(1 - \sin^2\frac{t}{2} \right) =$$
$$= -\cos\frac{t}{2} \cdot \cos^2\frac{t}{2}$$
$$= -\cos^3\frac{t}{2}$$

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$$\begin{vmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{vmatrix} = \sin^2 t + \cos^2 t = 1$$

$$\dot{c}(t) \times \ddot{c}(t) = a^2 \left(\frac{1}{2}\sin\frac{t}{2}(\cos t + 2), -\cos^3\frac{t}{2}, 1 \right)$$

$$\|\dot{c}(t) \times \ddot{c}(t)\|^2 = a^4 \left(\left(\frac{1}{2}\sin\frac{t}{2}(\cos t+2)\right)^2 + \left(-\cos^3\frac{t}{2}\right)^2 + 1 \right)$$

$$= a^4 \left(\frac{1}{4}\sin^2\frac{t}{2}(\cos t+2)^2 + \cos^6\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\frac{1}{4}\sin^2\frac{t}{2}\left(2\cos^2\frac{t}{2}+2-1\right)^2 + \cos^6\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\frac{1}{4}\sin^2\frac{t}{2}\left(2\cos^2\frac{t}{2}+1\right)^2 + \cos^6\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\frac{1}{4}\sin^2\frac{t}{2}\left(4\cos^4\frac{t}{2}+4\cos^2\frac{t}{2}+1\right) + \cos^6\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\sin^2\frac{t}{2}\cos^4\frac{t}{2} + \sin^2\frac{t}{2}\cos^2\frac{t}{2} + \frac{1}{4}\sin^2\frac{t}{2} + \cos^6\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\cos^4\frac{t}{2}\left(\sin^2\frac{t}{2} + \cos^2\frac{t}{2}\right) + \sin^2\frac{t}{2}\cos^2\frac{t}{2} + \frac{1}{4}\sin^2\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\cos^4\frac{t}{2} + \sin^2\frac{t}{2}\cos^2\frac{t}{2} + \frac{1}{4}\sin^2\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\cos^2\frac{t}{2}\left(\cos^2\frac{t}{2} + \sin^2\frac{t}{2}\right) + \frac{1}{4}\sin^2\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\cos^2\frac{t}{2} + \frac{1}{4}\sin^2\frac{t}{2} + 1 \right)$$

$$= a^4 \left(\frac{1+\cos t}{2} + \frac{1}{4}\cdot\frac{1-\cos t}{2} + 1 \right)$$

$$= \frac{a^4}{8}(4 + 4\cos t + 1 - \cos t + 8)$$

$$= \frac{a^4}{8}(13 + 3\cos t)$$

$$\Rightarrow \|\dot{c}(t) \times \ddot{c}(t)\| = \frac{a^2}{2\sqrt{2}}\sqrt{13 + 3\cos t}$$

$$\|\dot{c}(t)\|^2 = (-a\sin t)^2 + (a\cos t)^2 + \left(a\cos\frac{t}{2}\right)^2$$

$$= a^2\sin^2 t + a^2\cos^2 t + a^2\cos^2\frac{t}{2}$$

$$= a^2(\sin^2 t + \cos^2 t) + a^2\cos^2\frac{t}{2}$$

$$= a^2 + a^2\cos^2\frac{t}{2}$$

$$= a^2\left(1 + \cos^2\frac{t}{2}\right)$$

$$= a^2\left(1 + \frac{1+\cos t}{2}\right)$$

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$$\Rightarrow \|\dot{c}(t)\|^2 = a^2\left(1 + \frac{1+\cos t}{2}\right)$$

$$= \frac{a^2}{2}(3+\cos t)$$

$$\Rightarrow \|\dot{c}(t)\| = \frac{a}{\sqrt{2}}\sqrt{3+\cos t}$$

$$k_1 = \frac{\|\dot{c}(t)\times\ddot{c}(t)\|}{\|\dot{c}(t)\|^3} = \frac{\frac{a^2}{2\sqrt{2}}\sqrt{13+3\cos t}}{\frac{a^3}{2\sqrt{2}}\sqrt{(3+\cos t)^3}} \Rightarrow$$

$$\Rightarrow k_1 = \frac{\sqrt{13+3\cos t}}{a\sqrt{(3+\cos t)^3}}$$

$$\det(\dot{c}(t), \ddot{c}(t), \dddot{c}(t)) = a^3\begin{vmatrix} -\sin t & -\cos t & \sin t \\ \cos t & -\sin t & -\cos t \\ \cos\frac{t}{2} & -\frac{1}{2}\sin\frac{t}{2} & -\frac{1}{4}\cos\frac{t}{2}\end{vmatrix} =$$

$$= a^3\left(-\sin t\begin{vmatrix} -\sin t & -\cos t \\ -\frac{1}{2}\sin\frac{t}{2} & -\frac{1}{4}\cos\frac{t}{2}\end{vmatrix} + \cos t\begin{vmatrix}\cos t & -\cos t \\ \cos\frac{t}{2} & -\frac{1}{4}\cos\frac{t}{2}\end{vmatrix} + \right.$$

$$\left. + \sin t\begin{vmatrix}\cos t & -\sin t \\ \cos\frac{t}{2} & -\frac{1}{2}\sin\frac{t}{2}\end{vmatrix}\right)$$

$$= a^3\left(-\sin t\left(\frac{1}{4}\sin t\cos\frac{t}{2} - \frac{1}{2}\sin\frac{t}{2}\cos t\right) + \cos t\left(-\frac{1}{4}\cos\frac{t}{2}\cos t + \right.\right.$$

$$\left.\left. + \cos t\cos\frac{t}{2}\right) + \sin t\left(-\frac{1}{2}\sin\frac{t}{2}\cos t + \sin t\cos\frac{t}{2}\right)\right)$$

$$= a^3\left(-\frac{1}{4}\sin^2 t\cos\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2}\sin t\cos t - \frac{1}{4}\cos\frac{t}{2}\cos^2 t + \right.$$

$$\left. + \cos^2 t\cos\frac{t}{2} - \frac{1}{2}\sin\frac{t}{2}\sin t\cos t + \sin^2 t\cos\frac{t}{2}\right)$$

$$= a^3\left(\frac{3}{4}\sin^2 t\cos\frac{t}{2} + \frac{3}{4}\cos^2 t\cos\frac{t}{2}\right)$$

$$= a^3\,\frac{3}{4}\cos\frac{t}{2}\left(\sin^2 t + \cos^2 t\right)$$

$$= \frac{3}{4}a^3\cos\frac{t}{2}$$

$$k_2 = \frac{\det(\dot{c}(t), \ddot{c}(t), \dddot{c}(t))}{\|\dot{c}(t)\times\ddot{c}(t)\|^2} = \frac{\frac{3}{4}a^3\cos\frac{t}{2}}{\frac{a^4}{8}(13+3\cos t)} = \frac{6\cos\frac{t}{2}}{a(13+3\cos t)}$$

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$$\Rightarrow \quad K_2 = \frac{6\cos\frac{t}{2}}{a(13+3\cos t)}$$

Reperul lui Frenet

Fie $\{e_1, e_2, e_3\}$ reperul lui Frenet asociat curbei

$$\dot{c}(t) = \left(-a\sin t, a\cos t, a\cos\frac{t}{2}\right)$$

$$\|\dot{c}(t)\| = \frac{a}{\sqrt{2}}\sqrt{3+\cos t}$$

$$\dot{c}(t)\times\ddot{c}(t) = a^2\left(\frac{1}{2}\sin\frac{t}{2}(\cos t+2), -\cos^3\frac{t}{2}, 1\right)$$

$$\|\dot{c}(t)\times\ddot{c}(t)\| = \frac{a^2}{2\sqrt{2}}\sqrt{13+3\cos t}$$

$$e_1(t) = \frac{\dot{c}(t)}{\|\dot{c}(t)\|} = \left(\frac{a\sqrt{3+\cos t}}{\sqrt{2}}\right)^{-1}\left(-a\sin t, a\cos t, a\cos\frac{t}{2}\right)$$

$$= \frac{a\left(-\sin t, \cos t, \cos\frac{t}{2}\right)}{a\sqrt{\frac{3+\cos t}{2}}} \Rightarrow$$

$$\Rightarrow e_1(t) = \frac{1}{\sqrt{\frac{3+\cos t}{2}}}\left(-\sin t, \cos t, \cos\frac{t}{2}\right)$$

$$e_3(t) = \frac{\dot{c}(t)\times\ddot{c}(t)}{\|\dot{c}(t)\times\ddot{c}(t)\|}$$

$$= \left(\frac{a^2}{2\sqrt{2}}\sqrt{13+3\cos t}\right)^{-1} a^2\left(\frac{1}{2}\sin\frac{t}{2}(\cos t+2), -\cos^3\frac{t}{2}, 1\right)$$

$$= \frac{1}{a^2}\cdot\sqrt{\frac{8}{13+3\cos t}}\cdot a^2\left(\frac{1}{2}\sin\frac{t}{2}(\cos t+2), -\cos^3\frac{t}{2}, 1\right)$$

$$\Rightarrow e_3(t) = \sqrt{\frac{8}{13+3\cos t}}\left(\frac{1}{2}\sin\frac{t}{2}(\cos t+2), -\cos^3\frac{t}{2}, 1\right)$$

$$e_2(t) = e_3(t)\times e_1(t) \Rightarrow$$

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$$\Rightarrow e_3 = \sqrt{\frac{2}{3+\cos t}} \cdot \sqrt{\frac{8}{13+3\cos t}} \begin{vmatrix} \frac{1}{2}\sin\frac{t}{2}(\cos t+2) & -\sin t & i \\ -\cos^3\frac{t}{2} & \cos t & j \\ 1 & \cos\frac{t}{2} & k \end{vmatrix}$$

$$\begin{vmatrix} -\cos^3\frac{t}{2} & \cos t \\ 1 & \cos\frac{t}{2} \end{vmatrix} = -\cos^4\frac{t}{2} - \cos t$$

$$= -\cos^2\frac{t}{2} \cdot \frac{1+\cos t}{2} - \cos t =$$

$$= -\cos^2\frac{t}{2} \cdot \frac{1}{2} - \frac{1}{2}\cos^2\frac{t}{2}\cos t - \cos t =$$

$$= -\frac{1}{2}\cos^2\frac{t}{2} - \cos t\left(\frac{1}{2}\cos^2\frac{t}{2} + 1\right)$$

$$= -\frac{1}{2} \cdot \frac{1+\cos t}{2} - \cos t\left(\frac{1}{2}\ \frac{1+\cos t}{2} + 1\right)$$

$$= -\frac{1}{4} - \frac{\cos t}{4} - \frac{1}{2}\cos t \cdot \frac{1}{2}(1+\cos t) - \cos t$$

$$= -\frac{1}{4} - \frac{\cos t}{4} - \frac{1}{4}\cos t - \frac{1}{4}\cos^2 t - \cos t$$

$$= -\frac{1}{4}\cos^2 t - \frac{1}{4} - \frac{6\cos t}{4} =$$

$$= -\frac{3}{2}\cos t - \frac{1}{4}(\cos^2 t + 1)$$

$$\begin{vmatrix} \frac{1}{2}\sin\frac{t}{2}(\cos t+2) & -\sin t \\ 1 & \cos\frac{t}{2} \end{vmatrix} = \frac{1}{2}\sin\frac{t}{2}\cos\frac{t}{2}(\cos t+2) + \sin t$$

$$= \frac{1}{4}\sin t(\cos t+2) + \sin t$$

$$= \frac{\sin t\cos t}{4} + \frac{1}{2}\sin t + \sin t$$

$$= \frac{1}{4}\sin t\cos t + \frac{3}{2}\sin t$$

$$\begin{vmatrix} \frac{1}{2}\sin\frac{t}{2}(\cos t+2) & -\sin t \\ -\cos^3\frac{t}{2} & \cos t \end{vmatrix} = +\frac{1}{2}\sin\frac{t}{2}\cos t(\cos t+2) - \sin t\cos^3\frac{t}{2}$$

$$= +\frac{1}{2}\sin\frac{t}{2}(\cos t+2)\cos t - 2\sin\frac{t}{2}\cos^4\frac{t}{2}$$

$$= +\frac{1}{2}\sin\frac{t}{2}\cos^2 t + \sin\frac{t}{2}\cos t - 2\sin\frac{t}{2} \cdot \cos^4\frac{t}{2}$$

$$= +\frac{1}{2}\sin\frac{t}{2}\left(2\cos^2\frac{t}{2} - 1\right)^2 + \sin\frac{t}{2}\cos t - 2\sin\frac{t}{4}\cos^4\frac{t}{2}$$

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$$= \frac{1}{2}\sin\frac{t}{2}\left(4\cos^4\frac{t}{2} - 4\cos^2\frac{t}{2} + 1\right) + \sin\frac{t}{2}\cos t - 2\sin\frac{t}{2}\cos^4\frac{t}{2}$$

$$= 2\sin\frac{t}{2}\cos^4\frac{t}{2} - 2\sin\frac{t}{2}\cos^2\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2} +$$

$$+ \sin\frac{t}{2}\left(2\cos^2\frac{t}{2} - 1\right) - 2\sin\frac{t}{2}\cos^4\frac{t}{2}$$

$$= -2\sin\frac{t}{2}\cos^2\frac{t}{2} + \frac{1}{2}\sin\frac{t}{2} + 2\sin\frac{t}{2}\cos^2\frac{t}{2} - \sin\frac{t}{2}$$

$$= -\frac{1}{2}\sin\frac{t}{2}$$

$$\Rightarrow e_2 = \sqrt{\frac{16}{(3+\cos t)(13+3\cos t)}}\left(-\frac{3}{4}\cos t - \frac{1}{4}(\cos^2 t + 1), -\frac{1}{4}\sin t\cos t - \frac{3}{2}\sin t, -\frac{1}{2}\sin\frac{t}{2}\right)$$

Ecuațiile muchiilor și fețelor triedrului lui Frenet

Ecuația tangentei:

$$c(t) = \left(a + a\cos t, a\sin t, 2a\sin\frac{t}{2}\right)$$

$$\dot{c}(t) = \left(-a\sin t, a\cos t, a\cos\frac{t}{2}\right)$$

$$(tg): \quad \frac{X - x(t)}{\dot{x}(t)} = \frac{Y - y(t)}{\dot{y}(t)} = \frac{Z - z(t)}{\dot{z}(t)}$$

$$\frac{X - a(1+\cos t)}{-a\sin t} = \frac{Y - a\sin t}{a\cos t} = \frac{Z - 2a\sin\frac{t}{2}}{a\cos\frac{t}{2}}$$

Ecuația planului normal

$$\dot{x}(t)(X - x(t)) + \dot{y}(t)(Y - y(t)) + \dot{z}(t)(Z - z(t)) = 0$$

$$(-a\sin t)(X - a(1+\cos t)) + a\cos t(Y - a\sin t) + a\cos\frac{t}{2}\left(Z - 2a\sin\frac{t}{2}\right) = 0$$

$$-a\sin t X + a^2\sin t + a^2\sin t\cos t + a\cos Y - a^2\sin t\cos t +$$

$$+ a\cos\frac{t}{2} Z - 2a^2\sin\frac{t}{2}\cos\frac{t}{2} = 0$$

$$-a\sin X + a\cos Y + a\cos\frac{t}{2} Z + a^2\sin t - a^2\sin t = 0$$

$$\Rightarrow (\Pi_n): \ -a\sin X + a\cos Y + a\cos\frac{t}{2} Z = 0$$

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Ecuația binormalei

$$\frac{X-x(t)}{A(t)}=\frac{Y-y(t)}{B(t)}=\frac{Z-z(t)}{C(t)}$$

$$A(t)=\begin{vmatrix}\dot y(t) & \dot z(t)\\ \ddot y(t) & \ddot z(t)\end{vmatrix}\quad B(t)=-\begin{vmatrix}\dot x(t) & \dot z(t)\\ \ddot x(t) & \ddot z(t)\end{vmatrix}\quad C(t)=\begin{vmatrix}\dot x(t) & \dot y(t)\\ \ddot x(t) & \ddot y(t)\end{vmatrix}$$

$$A(t)=-\begin{vmatrix}-a\sin t & -\frac{a}{2}\sin\frac{t}{2}\\ a\cos t & a\cos\frac{t}{2}\end{vmatrix}=-\left(-a^2\sin t\cos\frac{t}{2}+\frac{a^2}{2}\sin\frac{t}{2}\cos t\right)$$

$$=-\left(-a^2\cdot 2\sin\frac{t}{2}\cos\frac{t}{2}\cos\frac{t}{2}+\frac{a^2}{2}\cdot\sin\frac{t}{2}\left(2\cos^2\frac{t}{2}-1\right)\right)$$

$$=-\left(-2a^2\sin\frac{t}{2}\cos^2\frac{t}{2}+a^2\sin\frac{t}{2}\cos^2\frac{t}{2}-\frac{a^2}{2}\sin\frac{t}{2}\right)$$

$$=-\left(-a^2\sin\frac{t}{2}\cos^2\frac{t}{2}-\frac{a^2}{2}\sin\frac{t}{2}\right)$$

$$=a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)$$

$$B(t)=-\begin{vmatrix}-a\sin t & a\cos\frac{t}{2}\\ -a\cos t & -\frac{a}{2}\sin\frac{t}{2}\end{vmatrix}=-\left(\frac{a^2}{2}\sin\frac{t}{2}\sin t+a^2\cos\frac{t}{2}\cos t\right)$$

$$=-\left(\frac{a^2}{2}\cdot 2\sin\frac{t}{2}\cdot\sin\frac{t}{2}\cos\frac{t}{2}+a^2\cos\frac{t}{2}\left(1-2\sin^2\frac{t}{2}\right)\right)$$

$$=-\left(a^2\sin^2\frac{t}{2}\cos\frac{t}{2}+a^2\cos\frac{t}{2}-2a^2\sin^2\frac{t}{2}\cos\frac{t}{2}\right)$$

$$=-\left(-a^2\sin^2\frac{t}{2}\cos\frac{t}{2}+a^2\cos\frac{t}{2}\right)$$

$$=a^2\sin^2\frac{t}{2}\cos\frac{t}{2}-a^2\cos\frac{t}{2}$$

$$=a^2\cos\frac{t}{2}\left(\sin^2\frac{t}{2}-1\right)$$

$$=-a^2\cos\frac{t}{2}\cdot\cos^2\frac{t}{2}$$

$$=-a^2\cos^3\frac{t}{2}$$

$$C(t)=\begin{vmatrix}-a\sin t & a\cos t\\ -a\cos t & -a\sin t\end{vmatrix}=a^2\sin^2 t+a^2\cos^2 t$$
$$=a^2$$

$$\Rightarrow\quad \frac{X-a(1+\cos t)}{a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)}=\frac{Y+a^2\sin t}{-a^2\cos^3\frac{t}{2}}=\frac{Z-2a\sin\frac{t}{2}}{a^2}$$

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Ecuatia planului osculator

$$A(t)(X-x(t))+B(t)(Y-y(t))+C(t)(Z-z(t))=0$$

$$a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)\left(X-a(1+\cos t)\right)+\left(-a^2\cos^3\frac{t}{2}\right)\left(Y-a\sin t\right)+$$
$$+\ a^2\left(Z-2a\sin\frac{t}{2}\right)=0$$

$$a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)X-a^3\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)(1+\cos t)+$$
$$+\left(-a^2\cos^3\frac{t}{2}\right)Y+a^3\cos^3\frac{t}{2}\sin t+$$
$$+\ a^2Z-2a^3\sin\frac{t}{2}=0$$

$$a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)X-a^2\cos^3\frac{t}{2}Y+a^2Z-$$
$$-\ a^3\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)\left(1+2\cos^2\frac{t}{2}-1\right)+a^3\cos^3\frac{t}{2}\cdot 2\sin\frac{t}{2}\cos\frac{t}{2}-$$
$$-\ 2a^3\sin\frac{t}{2}=0$$

$$a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)X-a^2\cos^3\frac{t}{2}Y+a^2Z-$$
$$-\ 2a^3\sin\frac{t}{2}\cos^4\frac{t}{2}-2a^3\sin\frac{t}{2}\cos^2\frac{t}{2}+2a^3\sin\frac{t}{2}\cos^4\frac{t}{2}-$$
$$-\ 2a^3\sin\frac{t}{2}=0$$

$$a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)X-a^2\cos^3\frac{t}{2}Y+a^2Z-$$
$$-\ 2a^3\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)=0$$

$$\Rightarrow\ a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)X-a^2\cos^3\frac{t}{2}Y+a^2Z-2a^3\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right)=0$$

Ecuatia normalei principale

$$\frac{X-x(t)}{l(t)}=\frac{Y-y(t)}{m(t)}=\frac{Z-z(t)}{n(t)}$$

$$l(t)=\begin{vmatrix} B(t) & C(t)\\ \dot y(t) & \dot z(t)\end{vmatrix}\quad m(t)=\begin{vmatrix} A(t) & C(t)\\ \dot x(t) & \dot z(t)\end{vmatrix}$$

$$n(t)=\begin{vmatrix} A(t) & B(t)\\ \dot x(t) & \dot y(t)\end{vmatrix}$$

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$$l(t) = \begin{vmatrix} -a^2\cos^3\frac{t}{2} & a^2 \\ a\cos t & a\cos\frac{t}{2} \end{vmatrix} = -a^3\cos^4\frac{t}{2} - a^3\cos t$$

$$= -a^3\left(\cos^4\frac{t}{2} + \cos t\right)$$

$$m(t) = -\begin{vmatrix} a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right) & a^2 \\ -a\sin t & a\cos\frac{t}{2} \end{vmatrix} =$$

$$= -\left(a^3\sin\frac{t}{2}\cos\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right) + a^3\sin t\right)$$

$$= -\left(a^3\sin\frac{t}{2}\cos\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right) + 2a^3\sin\frac{t}{2}\cos\frac{t}{2}\right)$$

$$= -a^3\sin\frac{t}{2}\cos\frac{t}{2}\left(\cos^2\frac{t}{2}+3\right)$$

$$n(t) = \begin{vmatrix} a^2\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right) & -a^2\cos^3\frac{t}{2} \\ -a\sin t & a\cos t \end{vmatrix} =$$

$$= a^3\sin\frac{t}{2}\cos t\left(\cos^2\frac{t}{2}+1\right) - a^3\sin t\cos^3\frac{t}{2}$$

$$= a^3\sin\frac{t}{2}\left(2\cos^2\frac{t}{2}-1\right)\left(\cos^2\frac{t}{2}+1\right) - 2a^3\sin\frac{t}{2}\cos^4\frac{t}{2}$$

$$= 2a^3\sin\frac{t}{2}\cos^2\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right) - a^3\sin\frac{t}{2}\left(\cos^2\frac{t}{2}+1\right) -$$

$$- 2a^3\sin\frac{t}{2}\cos^4\frac{t}{2}$$

$$= 2a^3\sin\frac{t}{2}\cos^4\frac{t}{2} + 2a^3\sin\frac{t}{2}\cos^2\frac{t}{2} - a^3\sin\frac{t}{2}\cos^2\frac{t}{2} -$$

$$- a^3\sin\frac{t}{2} - 2a^3\sin\frac{t}{2}\cos^4\frac{t}{2}.$$

$$= a^3\sin\frac{t}{2}\cos^2\frac{t}{2} - a^3\sin\frac{t}{2}$$

$$= a^3\sin\frac{t}{2}\left(\cos^2\frac{t}{2}-1\right)$$

$$= -a^3\sin\frac{t}{2}\cdot\sin^2\frac{t}{2}$$

$$= -a^3\sin^3\frac{t}{2}$$

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$$\frac{X-x(t)}{l(t)}=\frac{Y-y(t)}{m(t)}=\frac{Z-z(t)}{n(t)}$$

$$\Rightarrow \frac{X-a(1+\cos t)}{-a^3\left(\cos^4\frac{t}{2}+\cos t\right)}=\frac{Y-a\cos t}{-a^3\sin\frac{t}{2}\cos\frac{t}{2}\left(\cos^2\frac{t}{2}+3\right)}=\frac{Z-2a\sin\frac{t}{2}}{-a^3\sin^3\frac{t}{2}}$$

Ecuatia planului rectificant

$$l(t)(X-x(t))+m(t)(Y-y(t))+n(t)(Z-z(t))=0$$

$$\Rightarrow -a^3\left(\cos^4\frac{t}{2}+\cos t\right)\left(X-a(1+\cos t)\right)-$$
$$-a^3\sin\frac{t}{2}\cos\frac{t}{2}\left(Y-a\cos t\right)-$$
$$-a^3\sin^3\frac{t}{2}\left(Z-2a\sin\frac{t}{2}\right)=0$$

$$\Rightarrow -a^3\left(\cos^4\frac{t}{4}+\cos t\right)X-a^3\sin\frac{t}{2}\cos\frac{t}{2}Y-a^3\sin^3\frac{t}{2}Z$$
$$+a^4\left(\cos^4\frac{t}{2}+\cos t\right)(1+\cos t)+a^4\sin\frac{t}{2}\cos\frac{t}{2}\cos t+$$
$$+2a^4\sin^4\frac{t}{2}=0$$

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Curba este elice sau nu ?

$$\frac{K_1}{K_2} = \text{constant}$$

$$K_1 = \frac{\sqrt{13+3\cos t}}{a\sqrt{(3+\cos t)^3}}$$

$$K_2 = \frac{6\cos\frac{t}{2}}{a(13+3\cos t)}$$

$$\frac{K_1}{K_2} = \frac{\sqrt{13+3\cos t}}{a\sqrt{(3+\cos t)^3}} \cdot \frac{a(13+3\cos t)}{6\cos\frac{t}{2}}$$

$$= \sqrt{\left(\frac{13+3\cos t}{3+\cos t}\right)^3} \cdot \frac{1}{6\cos\frac{t}{2}}$$

$\neq$ constantă

$\Rightarrow c(t)$ nu poate fi o elice.

Lungimea unui arc al curbei

$$L(c/_{[a,b]}) = \int_a^b \|\dot{c}(t)\|\, dt$$

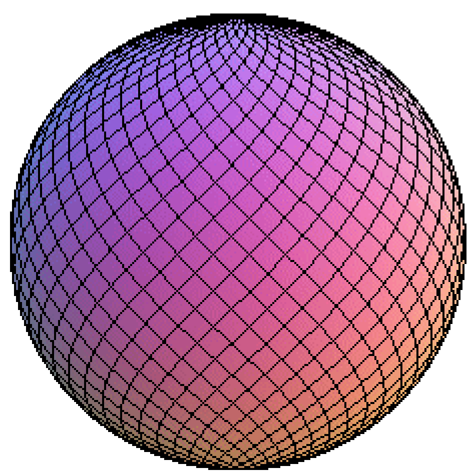
$$\Rightarrow L(c/_{[a,b]}) = \int_a^b \frac{a}{\sqrt{2}}\sqrt{3+\cos t}\, dt =$$

$$= \frac{a}{\sqrt{2}}\int_a^b \sqrt{3+\cos t}\, dt =$$

Exemplu concret

Un exemplu concret in care a fost folosita curba lui Viviani este “Muzeul maritim din Osaka” construit dupa schitele arhitectului P. Andreu

Constructia lui este realizata dupa o sfera acoperita de o retea de curbe Viviani.



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Anexa